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Lightness, heaviness and gravity

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Abstract

The gravity $g(H, \mathscr{H})$ of a graph H in the family of graphs $\mathscr{H}$ is the greatest integer n with the property that for every integer m, there exists a supergraph $G \in \mathscr{H}$ of H such that each subgraph of G, which is isomorphic to H, contains at least n vertices of degree $\geqslant m$ in G. We study the basic properties of the gravity function for various families of plane graphs. We also introduce and study the almost-light graphs and the absolutely heavy graphs. The paper concludes with few open problems.

Keywords: Gravity; Light graph; Heavy graph

1. Introduction

Throughout this paper, we consider connected graphs without loops or multiple edges. Let $\mathscr{H}$ be a family of graphs, and let H be a connected graph such that infinitely many members of $\mathscr{H}$ contain a subgraph isomorphic to H. Let $\varphi(H, \mathscr{H})$ be the smallest integer with the property that each graph $G \in \mathscr{H}$ which contains a subgraph isomorphic to H, contains also a subgraph $K \cong H$ such that, for every vertex $v \in K$,

$$d_G(v) \leqslant \varphi(H, \mathscr{H}).$$

If such a finite $\varphi(H, \mathscr{H})$ does not exist, we write $\varphi(H, \mathscr{H}) = +\infty$. We say that the graph H is *light* in the family $\mathscr{H}$ if $\varphi(H, \mathscr{H}) < +\infty$, otherwise we call it *heavy*. Thus, H is heavy in $\mathscr{H}$ if, for every integer m, there is a graph $G \in \mathscr{H}$ such that each isomorphic copy of H in G contains a vertex of degree $\geqslant m$ in G. The set of all light graphs in the family $\mathscr{H}$ is denoted by $\mathscr{L}(\mathscr{H})$.

The above definition of the lightness was firstly introduced in [7], but the notion appears in [3] (see also particular results in [1,5,4,10,9,12]). The article [8] gives a survey of results for various families of plane graphs.

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On the other hand, the heavy graphs were not studied. In this paper, based on the above definition of a heavy graph, we introduce the following measure of heaviness of graphs: the *gravity* $g(H, \mathscr{H})$ of a connected graph H in the family $\mathscr{H}$ of planar graphs is the greatest integer n with the property that for every integer m there exists a supergraph $G \in \mathscr{H}$ of H (that is, a graph $G \in \mathscr{H}$ such that H is a subgraph of G) such that each isomorphic copy of H in G contains at least n vertices of degree $\geqslant m$ in G. Hence, a graph is light in a family of graphs if and only if its gravity is zero. A graph whose gravity in a family is equal to n is called *n-heavy*.

In general, one can determine the gravity of particular graphs also for families of nonplanar graphs; note, however, that if a family $\mathscr{H}$ contains complete graphs of arbitrarily big order, then the gravity of every graph G is trivially equal to the number of its vertices.

When considering the gravity $g(H, \mathscr{H})$, we always assume that H is contained in infinitely many graphs of $\mathscr{H}$. Under this assumption, a nice property of gravity is the following observation:

(O) *If* $\mathscr{H}_2 \subseteq \mathscr{H}_1$, *then* $g(H, \mathscr{H}_2) \leqslant g(H, \mathscr{H}_1)$.

By (O), gravity is monotone with respect to inclusion of the families of graphs. On the other hand, it is not monotone with respect to taking subgraphs since, e.g., in the family of all polyhedral graphs and all the stars the 4-path is light (see [3]), but the 3-path is 1-heavy.

For proving the lightness of a graph in a given family of graphs, usually, the discharging method is used; for proving the heaviness, a construction of particular plane graphs is used. To determine the gravity of a graph, both these techniques are involved.

By P_k we denote the k-path, i.e., the path on k vertices and by C_k and S_k the k-cycle and the k-star $K_{1,k}$, respectively. The symbol $d(v)$ is used for the degree of a vertex v; a vertex of degree k is called a k-vertex. A vertex is called *big* or *small* with respect to a given (large enough) positive integer m if it is of degree $\geqslant m$ or $< m$, respectively. The minimum degree of a graph G is denoted by $\delta(G)$. By $\mathscr{P}_d$ we denote the family of planar graphs with the minimum vertex degree $\geqslant d$, and by $\mathscr{P}_d(w)$ the family of planar graphs with minimum vertex degree $\geqslant d$ and the minimum edge-weight (that is, the minimal sum of degrees of the endvertices of an edge in the graph) $\geqslant w$. The family of all 3-connected plane graphs is denoted by $\wp$. Given a plane graph G, the symbol $d(f)$ is used for the size of a face f of G (that is, the length of the boundary walk of f).

In the following two paragraphs, we describe some constructions which will be used to determine the gravity of some particular graphs.

Let G be a graph, and let v be a vertex of G. Take m vertex-disjoint copies of G and identify all the counterparts of v. The new graph is called an (m, G, v)-*star*. If G is a vertex-transitive graph, then we use to say an (m, G)-*star*. The vertex of identification is the *center* of the star. Thus, the (m, K_2)-star is just the m-star. Denote by $T_{m,h}$ the complete m-ary tree of height h. So, $T_{m,1}$ is the m-star. The vertices of degree 1 are called *leafs* and the highest vertex (that is, the vertex whose distances to all leafs are equal) is the *root*. If we identify each leaf of $T_{m,h}$ with the vertex v of a copy of G, the resulting graph is denoted by $T_{m,h}(G, v)$. Moreover, if G is a vertex-transitive graph or G is an (m, G_0)-star for some graph G_0 with v being the center of that star, then we write $T_{m,h}(G)$ instead of $T_{m,h}(G, v)$. Notice that $T_{m,h+1} = T_{m,h}(K_{1,m})$.

Let G be a connected plane graph on at least three vertices and let u, v be two distinct vertices lying on the border of the outerface of G. We use to say that the triple (G, u, v) is a *slice* and u, v are the *poles* of this slice. For the sake of the simplicity, we use to write for the slice (G, u, v) just G, when u, v are clear from the context. By the $(G, u, v; n)$-*melon* (or simply, *melon*) we denote the graph constructed in the following way: take n copies (slices) of G, identify all vertices corresponding to u into a new vertex and identify all vertices corresponding to v into another new vertex in all copies. In addition, if u and v are adjacent in G, then delete the multiple edges in the melon in order to obtain a simple graph. Two vertices resulted from this identification are called also the *poles* of the melon, the graph $G - u - v$ is the *pulp* of the slice (G, u, v). Observe that the melons are always planar graphs.

In each proof in this paper, based on the discharging method, we consider a hypothetical counterexample G with vertex set $V(G)$ and face set $F(G)$. We assign an initial charge c to every vertex $v \in V(G)$ and every face $f \in F(G)$ of the graph G in the following way:

$$c(v) = \alpha\, d(v) - 6 \quad \text{and} \quad c(f) = (3 - \alpha)\, d(f) - 6, \tag{1}$$

where α is some prescribed number. It follows from the Euler's formula that the total sum of the charge of the vertices and the faces of G is equal to -12 according to the assignment by (1). We will redistribute the charge of the vertices and the faces of G by applying certain rules without changing the total sum of all charges. Denote by $c^*(x)$ the charge of a vertex or a face x after applying these rules (the *final charge* of x). In the proof of each claim, we will prove that each face and each vertex of G has a nonnegative final charge, which gives a contradiction.

2. Light stars and heavy paths

In this section, we study the gravity of the paths and the stars in the families $\mathscr{P}_d$. It is shown that the stars are either light or 1-heavy. And, for the paths, it is shown that their gravity is close to their length. We first prove the following lemma:

Lemma 2.1. *Let $G \in \mathscr{P}_d$ be a planar graph with precisely b vertices of degree strictly greater than d, where $d \in \{1, \dots, 5\}$. Then, $g(G, \mathscr{P}_d) \geqslant b$.*

Proof. Identify each vertex of G of degree $\geqslant d + 1$ with the center of an (m, S)-star, where $S := K_{d+1}$ for $d = 1, 2, 3$, $S := O$ (octahedron) for $d = 4$, and $S := I$ (icosahedron) for $d = 5$. Denote the resulting graph by G^* and observe that it is a graph from $\mathscr{P}_d$. Notice that every copy of G in G^* contains all b big vertices of G. Now, the proof easily follows. $\square$

2.1. The gravity of stars in $\mathscr{P}_d$

Proposition 2.2. *Let $s_1 = s_2 = 0$, $s_3 = 1$, $s_4 = 2$, $s_5 = 4$ and $d \in \{1, \dots, 5\}$. If $k \leqslant s_d$ then the star S_k is light in $\mathscr{P}_d$, and otherwise it is 1-heavy in $\mathscr{P}_d$.*

Proof. We will show first that the gravity of each star in $\mathscr{P}_d$ is at most 1. Suppose that, for fixed integer k and large enough integer m, there exists a graph $G \in \mathscr{P}_d$ which contains at least one k-star as a subgraph, and every such k-star contains at least two big vertices. Note that in that case G has at least two big vertices. Moreover, every big vertex of G has at least $m - k + 1$ big neighbors; otherwise, we encounter a k-star with only one big vertex. Now, consider the subgraph M induced by the big vertices of G. Then, $\delta(M) \geqslant m - k + 1$. But, M is planar, so it contains a vertex of degree at most 5 and since m is large enough, it is a contradiction.

Now, we consider several cases regarding d. Since $\mathscr{L}(\mathscr{P}_1) = \mathscr{L}(\mathscr{P}_2) = \{P_1\}$, it follows that the 0-star S_0 $(= P_1)$ is the only light star in $\mathscr{P}_1$ and $\mathscr{P}_2$. Hence, $s_1 = s_2 = 0$. For $d = 3$, we have $\mathscr{L}(\mathscr{P}_3) = \{P_1, P_2\}$, and thus we have that $s_3 = 1$ is the right number. In [11] it is shown that $\mathscr{L}(\mathscr{P}_4) = \{P_1, P_2, P_3, P_4\}$. Thus, S_0, S_1, S_2 are the only light stars in $\mathscr{P}_4$ and $s_4 = 2$. For $\mathscr{P}_5$, the set of light graphs $\mathscr{L}(\mathscr{P}_5)$ is not known, but in [6] it is proven that the k-star is heavy in $\mathscr{P}_5$ if and only if $k \geqslant 5$. Thus, $s_5 = 4$. $\square$

2.2. The gravity of paths in $\mathscr{P}_1$

Theorem 2.3.

$$g(P_n, \mathscr{P}_1) = \begin{cases} n - 2, & n = 3, 5, \\ n - 1 & \text{otherwise.} \end{cases}$$

Proof. If $n = 3$ then the proof follows by Proposition 2.1. Suppose now that $n = 5$. By Lemma 2.1, $g(P_5, \mathscr{P}_1) \geqslant 3$. In order to prove the equality, suppose that G is a planar graph with at least one 5-path, and every such path contains at least four big vertices. Now, color in G each small vertex with respect to a given integer m by 0 and color each big vertex having at least two small neighbors by color 1. The remaining big vertices of G color by 2.

If G contains a 3-path which vertices are consecutively colored by 1, 2, 1 or 1, 1, 1, then we can easily find a 5-path in G with both endvertices being small. Otherwise, there is no vertex colored by 2 or every vertex colored by 2 has at most one neighbor colored by 1. In the first case, G must contain a 3-path with all vertices colored by 1, which is a contradiction. In the second case consider the subgraph of G induced by the vertices colored by 2. Note that this graph has minimum degree $m - 2$. This implies that G is not planar for $m \geqslant 8$, a contradiction.

Finally, suppose that $n \neq 3, 5$. For n even, consider the complete m-ary tree of height $n/2$. In this graph, every n-path P_n contains $n-1$ big vertices. If n is odd then consider the graph constructed from a 4-path by identifying each endvertex with the root of a copy of $T_{m,\lfloor n/2\rfloor-1}$ and each of the two inner vertices of the 4-path identify with the center of a copy of the m-star. Observe that in this graph each n-path contains $n-1$ vertices of degree $\geqslant m$. Thus, $g(P_n, \mathscr{P}_1) \geqslant n-1$.

To show the equality $g(P_n, \mathscr{P}_1) = n-1$, assume that for each large enough integer m there exists a connected graph $G_m \in \mathscr{P}_1$ in which every n-path consists only of big vertices. Moreover, we assume that G_m has at least one n-path, say P. Then, P can be easily extended to a path P^* of length $\geqslant m$, consisting only of big vertices. If G_m contains a vertex x of degree $< m$, then, by connectedness of G_m, there exists an x–y-path with $y \in P^*$ and no other vertex of that path belongs to P^*. Now, this x–y-path can be extended (using the part of P^*, if necessary) to an n-path with less than n big vertices, a contradiction. Hence, each vertex of G_m has to be big, which contradicts the planarity of G_m. □

2.3. *The gravity of paths in $\mathscr{P}_2$*

Theorem 2.4. *For each $n \geqslant 4$, the gravity of the n-path P_n in the family $\mathscr{P}_2$ is at most $n-2$.*

Proof. Suppose that the theorem is false, and there exists an $n \geqslant 4$ such that for a large enough integer m, there exists a connected graph G with at least one n-path, and each its n-path has at least $n-1$ big vertices. For the sake of simplicity, an n-path with at least two small vertices is called *good*. Thus, the assumption is that G has no good n-path.

Claim 1. *G contains a path $P^* = y_1 y_2 \ldots y_{2n+1}$ such that y_{n+1} is a small vertex.*

By the assumptions, G contains a path on n vertices, say $P = x_{-s} \ldots x_{-1} x_0 x_1 x_2 \ldots x_{n-s-1}$. Since $\mathscr{P}_2 \subseteq \mathscr{P}_1$, by Theorem 2.3, we can assume that P has a small vertex, say x_0. We may assume that all other vertices of P are big.

Consider first the case that $s \neq 0$ and $n-s-1 \neq 0$. In this case both endvertices of P are big. In what follows, we will extend P in both directions to obtain the required path P^*. First, set $i := n-s$. Next, repeat the following procedure until $i > n$: choose a vertex which is a neighbor of x_{i-1} and which does not belong to P. This is possible since x_{i-1} is a big vertex and so it is adjacent to a vertex, which does not belong to P. Denote this vertex by x_i, and afterwards extend P by this vertex, i.e. set $P := P x_i$. Note that x_i is a big vertex, otherwise we obtain an n-subpath of P with two small vertices x_0 and x_i. Finally, set $i := i+1$.

We apply the above procedure also in the other direction in order to obtain the required path $P^* = x_{-n} x_{-n+1} \ldots x_{-1} x_0 x_1 x_2 \ldots x_n$.

Suppose now that $s = 0$. Then, $P = x_0 x_1 x_2 \ldots x_{n-1}$. If x_0 has a neighbor which is not on P, then it must be a big vertex. In this case, denote it by x_{-1} and consider the path $x_{-1} x_0 x_1 \ldots x_{n-2}$ as in the previous case in order to construct P^*. Otherwise, all neighbors of x_0 belong to P. Let x_l $(l > 1)$ be such a neighbor of x_0. In this case, set $P := x_{l-1} \ldots x_1 x_0 x_l x_{l+1} \ldots x_{n-1}$, and afterwards argue as in the first case of this claim. This establishes Claim 1.

Notice that, besides y_{n+1}, possible small vertices of P^* are y_1 and y_{2n+1}. And, all other vertices of P^* are big.

Claim 2. *G has no two adjacent small vertices.*

Suppose that the claim is false and suppose that u, v are two such vertices. Since the graph G is connected, it has a path Q with one endvertex in $\{u, v\}$, say u, and the other endvertex in $V(P^*)$, say y_p, where $p \in \{1, \ldots, 2n+1\}$. Notice that we did not exclude the possibility that one of the vertices u and v (or both) belong to P^*. We may assume that Q does not contain the vertex v, and it does not contain any other vertex of P^*. Now, it is easy to see that one of the two subgraphs $vQy_{p+1} \ldots y_{2n+1}$ and $vQy_{p-1} \ldots y_1$ contains a good n-path as a subgraph, a contradiction.

Claim 3. *G has no big vertex adjacent to two small vertices.*

Suppose that the claim is false and that a big vertex w is adjacent to two small vertices x_1 and x_2. By the previous claim, x_1 and x_2 are nonadjacent. If there is a path from x_1 to some vertex of P^*, which does not contain w and x_2, then we can easily find a path of length n which contains both x_1 and w. Similarly, we find a good n-path, if there exists a path from x_2 to a vertex of P^*, which does not contain x_1 and w. (Again, we do not exclude the possibility that some of x_1, x_2, w may belong to P^*.) Otherwise, each path with one endvertex in $\{x_1, x_2\}$ and other one in P^* must contain w. Thus, w is a cut-vertex of G, and it separates P^* from x_1 and x_2. In this case, let Q^* be a shortest path from w to P^*.

Suppose first that x_1 and x_2 belong to the same block of G. Let R be a shortest path between x_1, x_2 which does not contain w. Since x_1, x_2 are from the same block, the path R exists. If R is of length $\geqslant n-2$, then the cycle wR contains a good n-path. And, if R is of length smaller than $n-2$, then using a subpath of $Q^* \cup P^*$, the path R can be extended to a good n-path.

Now, we may assume that x_1 and x_2 belong to different blocks of G. Here, we argue similarly as above. Let R be a longest path in $G-w$, which contains x_2 as an endvertex. Denote by x_2^* the other endvertex of R. If R is of length $\geqslant n-3$, then the path x_1wR contains a good n-path. Otherwise, R is of length $<n-3$. By the choice of R it follows that all neighbors of x_2^* in $G-w$ belong to R. Hence, x_2^* is of degree $\leqslant n-3$ in $G-w$, and so it is of degree $\leqslant n-2$ in G. So, x_2^* is a small vertex. Hence, R can be extended to a good n-path with the vertices from $Q^* \cup P^*$. This proves the claim.

From the last claim, it follows that the planar graph, constructed from G by removing the small vertices, has a minimum degree at least $m-1$. But it is a contradiction and the end of the proof. □

We conclude this section with the following table for the gravity of the paths in $\mathscr{P}_2$. For $n=1, 2, 3$, the values follow by Proposition 2.2. So, assume that $n \geqslant 4$. Theorem 2.4 gives us that the gravity of the n-path is at most the value in the corresponding entry of the table. For $n=4, 5$ consider the graph $K_{2,m}$. For $n=6, 7, 8$ consider $T_{m,1}(K_{2,m}, v)$, where v is one of the two m-vertices in $K_{2,m}$. Let S be an (m, C_3)-star. For $n \geqslant 10$, construct the following graph: identify each endvertex of P_{n-6} with the root of a copy of $T_{m,2}(S)$ and identify each inner vertex of P_{n-6} with the center of a copy of S. Finally, for $n=9$ we use a similar construction: identify each endvertex of P_4 with the root of a copy of $T_{m,1}(S)$ and identify each inner vertex of P_4 with the center of a copy of S. In the so-constructed graph, each n-path contains $n-2$ big vertices.

The table leaves for the n-path P_n with $n \in \{5, 7, 8, 9\}$, the question whether its gravity is $n-3$ or $n-2$. The authors of this paper expect that the right lower bound is $n-3$.

2.3.1. *The gravity of paths in the subfamily of 2-connected graphs of $\mathscr{P}_2$*

Considering the lower bound of the gravity in Table 1, all presented graphs have the property that each big vertex is a cut-vertex, and so those graphs have many blocks. However, the next result shows that restricting to the family of 2-connected graphs of $\mathscr{P}_2$, call it $\mathscr{P}_2^*$, the gravity of infinitely many n-paths P_n is asymptotically close to n. Perhaps, the claim remains valid if we consider the realm of all paths and not only the infinite subfamily.

Proposition 2.5. *For infinitely many k, the k-path P_k has the gravity of order $k-\mathrm{o}(k)$ in $\mathscr{P}_2^*$.*

Proof. Let G_i be an $(S_i, u_i, v_i; m)$-melon, where $S_1 := P_3$ with poles being the vertices of degree 1, and S_i is obtained from G_{i-1} by joining the poles u_{i-1}, v_{i-1} of G_{i-1} with two new vertices u_i, v_i by edges $u_{i-1}u_i$ and $v_{i-1}v_i$.

Firstly, we will determine the length of a longest path L_i in G_i. Observe that L_i is contained in precisely three slices of G_i. Let $S_i^{(1)}$ be the slice where L_i starts, $S_i^{(2)}$ be the one that L_i passes through and $S_i^{(3)}$ be the one where L_i ends. Then L_i contains two poles of G_i, $2i-1$ vertices of the pulp of $S_i^{(2)}$, and i^2 vertices in each pulp of $S_i^{(1)}$ and $S_i^{(3)}$ (the first and last parts of L_i in $S_i^{(1)}$ and $S_i^{(3)}$ together lie in four slices $S_{i-1}^{(1)}$, $S_{i-1}^{(2)}$, $S_{i-1}^{(2)}$ and $S_{i-1}^{(3)}$ of two copies of G_{i-1}; then the result follows by induction). Thus, L_i has $2i(i+1)+1$ vertices.

Next, we will determine how many vertices of L_i are big. In $S_i^{(2)}$, L_i passes through exactly one 2-vertex; in each of $S_i^{(1)}$ and $S_i^{(3)}$, L_i passes through i 2-vertices. All other vertices of L_i are big. Thus, L_i contains $2i(i+1)+1-1-2i=2i^2$ big

Table 1
The gravity of n-paths in $\mathscr{P}_2$

n	1	2	3	4	5	6	7	8	9	$\geqslant 10$
$g(P_n, \mathscr{P}_2)$	0	1	1	2	2 or 3	4	4 or 5	5 or 6	6 or 7	$n-2$

vertices. Counting the ratio of big and all vertices of L_i, we obtain that it is $\geqslant 2i^2/(2i(i+1)+1)=1-(2i+1)/(2i^2+2i+1)$.

Consider a path L of length l in G_i where $2(i-1)i+1<l<2i(i+1)+1$. Then L is contained in two or three slices of G_i which separate it into two or three subpaths with not more than i 2-vertices. Hence, L contains at least $l-3i$ big vertices and the ratio big and all vertices of L gives $\geqslant (l-3i)/l=1-3i/l>1-3i/(2(i-1)i+1)=1-3/i=1-o(1)$ for l large enough.

From the estimations above, the claim follows. □

One can obtain similar asymptotical results also for the families of all 2-connected plane graphs of minimum degree $d\in\{3,4,5\}$. The construction is the same as in the proposition above, just taking $S_1:=K_4^-$ for $d=3$, $S_1:=O^-$ (an octahedron without an edge) for $d=4$, and $S_1:=I^-$ (an icosahedron without an edge); the poles of S_1 are the vertices of degree $d-1$.

3. Almost-light graphs

We say that a graph G is *almost-light* in the family $\mathscr{H}$ if it is 1-heavy and every its proper connected subgraph is light. Note that if every proper subgraph of a graph G is light in $\mathscr{H}$, then G is not necessarily 1-heavy, just consider the odd cycles and the family $\wp$ (see Theorem 4.1).

Let $\mathscr{AL}(\mathscr{G})$ be the set of all almost-light graphs in the family $\mathscr{G}$. Given a set X of graphs of a family $\mathscr{G}$, let $\overline{X}$ be the set of graphs of $\mathscr{G}$ such that for each graph $G\in\overline{X}$, every proper connected subgraph of G belongs to X. Now, we immediately obtain that $\mathscr{AL}(\mathscr{G})\subseteq\overline{\mathscr{L}(\mathscr{G})}$.

For $\mathscr{L}(\mathscr{G})$ finite, the set of heavy graphs in $\mathscr{G}$ may be infinite, as seen from Proposition 2.2. On the other hand, in this case only the finite number of graphs has to be examined for specifying the set $\mathscr{AL}(\mathscr{G})$. Note that for the families $\mathscr{P}_1$ and $\mathscr{P}_2$, the 2-path P_2 is the only almost-light graph. The next theorem describes the set of almost-light graphs in other three families of plane graphs, where the characterization of light graphs is complete (see [11]).

Theorem 3.1. *For the families of graphs $\mathscr{P}_3$, $\mathscr{P}_3(7)$, $\mathscr{P}_4$, it holds*

(a) $\mathscr{AL}(\mathscr{P}_3)=\{P_3\}$,
(b) $\mathscr{AL}(\mathscr{P}_3(7))=\{P_4,S_3\}$,
(c) $\mathscr{AL}(\mathscr{P}_4)=\{C_3,S_3,P_5\}$.

Proof. (a) We have $\mathscr{L}(\mathscr{P}_3)=\{P_1,P_2\}$ and $\overline{\mathscr{L}(\mathscr{P}_3)}=\{P_1,P_2,P_3\}$. Then, the result follows immediately by Proposition 2.2.

(b) We have $\mathscr{L}(\mathscr{P}_3(7))=\{P_1,P_2,P_3\}$ (see [2,11]) and $\overline{\mathscr{L}(\mathscr{P}_3(7))}=\{P_1,P_2,P_3,P_4,C_3,S_3\}$. By Proposition 2.2, gravity of S_3 in $\mathscr{P}_3$ and $\mathscr{P}_4$ is 1. Since $\mathscr{P}_4\subseteq\mathscr{P}_3(7)\subseteq\mathscr{P}_3$, by the observation (O) (from the introduction section) it follows that the gravity of S_3 in $\mathscr{P}_3$ is 1.

Next, we will show that $C_3\notin\mathscr{AL}(\mathscr{P}_3(7))$. Let S be the plane graph comprised of two 8-cycles $B=a_1b_1a_2b_2a_3b_3a_4b_4$, $C=a_1c_1a_2c_2a_3c_3a_4c_4$ and two vertices b, c such that $N(b)=\{b_1,\dots,b_4\}$, $N(c)=\{c_1,\dots,c_4\}$; further, let u,v be two adjacent vertices of the facial cycle $a_4b_4a_1c_4$ of S.

Let m be an integer. Consider a 3-cycle $C_3=x_1x_2x_3$ and, for each $i=1,2,3$ construct the $(S,u,v;m)$-melon G^i. Then identify the endvertices of each edge x_ix_{i+1} (index is taken modulo 3) with poles u^i,v^i (the counterparts of u,v in G^i). The resulting graph $\widetilde{G}$ belongs to $\mathscr{P}_3(7)$ and contains only one 3-cycle, all vertices of which are big (see Fig. 1). Thus $g(C_3,\mathscr{P}_3(7))=3$, which establishes the claim.

It remains to show that the gravity of P_4 in $\mathscr{P}_3(7)$ is 1. If it is not true, then for every integer m, there exists a graph $G_m\in\mathscr{P}_3(7)$ in which every 4-path P_4 contains at least two big vertices. Consider the initial charge assignment by (1) with $\alpha=\frac{3}{2}$ for $G:=G_m$, and the following discharging rules:

*Rule R*1: Each big vertex sends $\frac{3}{4}$ to each adjacent 3-vertex.

*Rule R*2: Each big vertex sends $3/2k$ to each incident triangular face f where $k\in\{1,2,3\}$ is the number of big vertices incident with f.

It is enough to deal only with 3-vertices, 3-faces, and big vertices. Each 3-vertex is adjacent with at least two big vertices (otherwise, it is adjacent with at least two vertices of degree $<m$ and, together, they can be extended to a

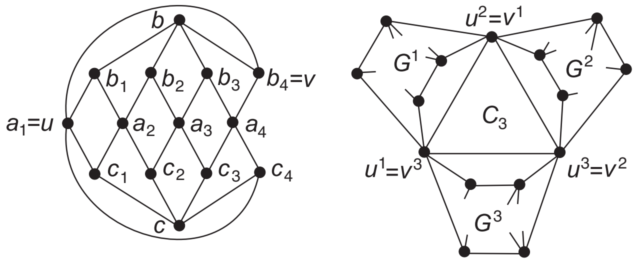

Fig. 1. The construction for showing that $g(C_3, \mathscr{P}_3(7)) = 3$.

4-path containing at most one big vertex, a contradiction). Hence, by R1, its final charge is nonnegative. Similarly, each triangular face is incident with at least one big vertex and, by R2, it has also nonnegative final charge.

Consider a big vertex x of degree $d \geqslant m$. If x is not incident with a triangular face, then $c^*(x) \geqslant \frac{3}{2}d - 6 - d \cdot \frac{3}{4} \geqslant 0$ for large m. If x is incident with a triangular face β having two remaining vertices of degree $< m$, then all remaining $d-2$ neighbors of x are big (otherwise, the vertices of β can be extended to a 4-path contradicting the assumptions on G). Thus, $c^*(x) \geqslant \frac{3}{2}d - 6 - \frac{3}{2} - 2 \cdot \frac{3}{4} - \frac{3}{6} \cdot (d-3) \geqslant 0$ for m large enough. So we can assume that every triangular face incident with x contains at least two big vertices. Denote by $x_1, \ldots, x_d$ the neighbors of x as they appear around x in a cyclic order, and by f_i the face which contains the subwalk $x_i x x_{i+1}$ (index modulo d). Without loss of generality, assume that there exist integers $r \geqslant 0, t_1, \ldots, t_r \geqslant 1$ and $i_2, \ldots, i_r$ such that $0 \leqslant t_1 + \cdots + t_r \leqslant d$, $1 + t_1 < i_2$, $i_{j-1} + t_{j-1} < i_j$ and $f_1, \ldots, f_{t_1}, f_{i_2}, \ldots, f_{i_2+t_2}, \ldots, f_{i_r}, \ldots, f_{i_r+t_r}$ are consecutive triangular faces; thus, among all the faces incident with x we specify the groups of consecutive triangular faces consisting of $t_1, \ldots, t_r$ triangles. Then $c^*(x) \geqslant \frac{3}{2}d - 6 - \frac{3}{4}(t_1 + \cdots + t_r) - \frac{3}{4}(d - (t_1+1) - \cdots - (t_r+1)) - \frac{3}{4}(\lceil t_1/2 \rceil - \cdots - \lceil t_r/2 \rceil) = \frac{3}{4}(d+r) - \frac{3}{4}(\lceil t_1/2 \rceil - \cdots - \lceil t_r/2 \rceil) \geqslant \frac{3}{8}r + \frac{3}{4}d - 6 \geqslant 0$ for d large enough. This completes the proof.

(c) We have $\mathscr{L}(\mathscr{P}_4) = \{P_1, P_2, P_3, P_4\}$ (see [11]) and $\overline{\mathscr{L}(\mathscr{P}_4)} = \{P_1, P_2, P_3, P_4, P_5, C_3, C_4, S_3\}$.

First, we show that $g(C_4, \mathscr{P}_4) \geqslant 2$. Fix an integer m and take the graph S of the Archimedean polytope of type $(3, 5, 3, 5)$. This graph can be obtained from the dodecahedron by cutting its vertices in such a way that the resulting graph is 4-regular. Let u, v be two nonadjacent vertices of S lying on the outer pentagonal face of S and let $\widetilde{G}$ be the $(S, u, v; m)$-melon. Now, take C_4 and identify endvertices of each its edge with poles of a copy of $\widetilde{G}$. It is easy to see that, in the resulting graph, every 4-cycle contains at least two vertices of degree $\geqslant m$.

Since C_3 is heavy in $\mathscr{P}_4$, the gravity $g(C_3, \mathscr{P}_4) \geqslant 1$. Assume $g(C_3, \mathscr{P}_4) \geqslant 2$. Then for each large enough m there is a graph $G_m \in \mathscr{P}_4$ such that every its triangle contains at least two big vertices. Consider the initial charge assignment by (1) with $\alpha = \frac{3}{2}$ and the following discharging rule:

*Rule R*1: Each big vertex sends $\frac{3}{4}$ to each incident triangular face.

It is enough to consider only the triangular faces and the big vertices. If f is a triangular face, it contains at least two big vertices, thus $c^*(f) \geqslant -\frac{3}{2} + 2 \cdot \frac{3}{4} = 0$. And, if v is a big vertex of degree d, then $c^*(v) \geqslant \frac{3}{2}d - 6 - \frac{3}{4}d \geqslant 0$. Thus, the proof is complete. □

In what follows we show that the gravity of P_5 is 1 in $\mathscr{P}_4$. So suppose that it is false. Then, for every large enough m there exists a graph $G_m \in \mathscr{P}_4$ such that every 5-path in G_m contains at least two big vertices. We say that two adjacent vertices u and v are *j-adjacent* ($j \in \{0, 1, 2\}$), if the edge uv is incident with precisely j triangular faces. Consider the initial charge assignment by (1) with $\alpha = 1$ and the following discharging rules:

*Rule R*1: Each face f of degree $\geqslant 4$ sends $\frac{1}{2}$ to each incident vertex of degree 4 or 5.

*Rule R*2: Each big vertex sends $\frac{1}{3}$ to each adjacent 5-vertex and $(2+j)/4$ to each j-adjacent 4-vertex.

It is easy to see that each face of G_m has even a positive final charge. Consider a 5-vertex x. If x is incident with at least two faces of degree $\geqslant 4$, then $c^*(x) \geqslant -1 + 2 \cdot \frac{1}{2} = 0$; otherwise, at least three of the neighbors of x are big and $c^*(x) \geqslant -1 + 3 \cdot \frac{1}{3} = 0$.

Now, let x be a 4-vertex. If x is not incident with a triangular face, then $c^*(x) \geqslant -2 + 4 \cdot \frac{1}{2} = 0$. Otherwise, x is adjacent with at least two big vertices and, as it can be checked in a routine manner, they send in total at least $\frac{l}{2}$ to x, where l is the number of triangular faces incident with x. Thus, the final charge of x is nonnegative.

Finally, let x be a big vertex of degree $d \geqslant m$ and let $x_1, \dots, x_d$ be its neighbors. We estimate how much of charge x sends to its neighbors in average. Consider the following averaging of the charges sent from x:

(A1) If a 4-vertex x_i receives 1 from x and $x' \in \{x_{i-1}, x_{i+1}\}$ is a neighbor of x receiving < 1 from x, then let x_i donate $\frac{1}{32}$ to x'.
(A2) If a 4-vertex x_i receives 1 from x and x_{i-1}, x_{i+1} are 4-vertices receiving 1 from x, then let x_i donate $\frac{1}{32}$ to each of x_{i-2}, x_{i+2}.

We will show that, after the averaging takes part, each neighbor x_i of x receives < 1 in average; since x is big, it implies that $c^*(x) \geqslant 0$ for m sufficiently large. To show this, several cases are considered.

Let x_i be a 4-vertex receiving 1 from x such that x_{i-1}, x_{i+1} are also 4-vertices receiving 1 from x; note that x_{i-2}, x_{i+2} are necessarily big. Then, by (A2), x_i preserves $1 - 2 \cdot \frac{1}{32} < 1$.

Let x_i be a 4-vertex receiving 1 from x such that—without loss of generality—x_{i-1} receives < 1 from x. Then, by (A1), x_i preserves $\leqslant 1 - \frac{1}{32} < 1$.

Finally, let x_i be a neighbor of x which receives < 1 from it; by R2, it receives $\leqslant \frac{3}{4}$. Observe that if all possible donations by (A1) and (A2) to x_i would apply, then still x_i would preserve $\leqslant \frac{3}{4} + 4 \cdot \frac{1}{32} = \frac{7}{8} < 1$. This establishes the proof. □

4. Absolutely heavy graphs

According to its definition, the gravity of a graph H in a given family $\mathscr{H}$ is bounded above by the number of its vertices. In the case of equality, we will say that H is *absolutely heavy* in $\mathscr{H}$. The following theorem shows that there are many absolutely heavy graphs in the families of graphs which are subjects of our consideration.

Theorem 4.1. (a) *Every planar graph of minimum degree greater than d is absolutely heavy in $\mathscr{P}_d$, where $d \in \{1, 2, 3, 4\}$.*

(b) *Every planar graph which is not a tree is absolutely heavy in $\mathscr{P}_1$.*

(c) *Infinitely many trees are absolutely heavy in $\mathscr{P}_1$.*

(d) *Every cycle is absolutely heavy in $\mathscr{P}_3$.*

(e) *Each of the cycles C_4, C_6, C_8, C_{10}, and C_n with n odd is absolutely heavy in $\wp$.*

Proof. For the purpose of the proof, let Q^-, D^-, O^- and K_4^- be the cube, the dodecahedron, the octahedron, and the complete graph K_4 minus one edge, respectively. If not stated otherwise, the endvertices of the deleted edge will be referred as u and v in this proof.

(a) The proof follows from Lemma 2.1.

(b) Let $G \in \mathscr{P}_1$. Let G' be the maximal subgraph of G, whose all vertices are of degree > 1, i.e. G' is a maximal (induced) subgraph of G with minimum degree at least 2. Then, the set $E(G) \backslash E(G')$ induces a set of trees. Each of these trees is considered as a rooted tree with the root being the unique common vertex with G'. Let h be the maximum of heights of these rooted trees. Construct the graph $\widetilde{G}$ in the following way: identify each vertex of G with the root of a copy of $T_{m,h+1}$. In $\widetilde{G}$, every vertex of each subgraph isomorphic to G is big.

(c) To show that there are trees which are absolutely heavy in $\mathscr{P}_1$, consider the k-star S_k with $k \geqslant 3$ with edges $e_1, \dots, e_k$. Now, subdivide each edge e_i with 10^x new vertices where x is a large enough integer; denote by T' the obtained tree and by $v_1, \dots, v_k$ the neighbors of leafs of T'. Now, identify each vertex v of T', $v \neq v_i, i = 1, \dots, k$ with the central vertex of a copy of the m-star; further, for each $i = 1, \dots, k$ identify the vertex v_i with the root of a copy of $T_{m,2}$. In the resulting graph, every isomorphic copy T'' of T' contains the center c of the original S_k. Observe that each path of length $10^x + 1$ with endvertex c consists only of big vertices; we can conclude that every vertex of T'' is big, which proves the claim. Notice that this construction does not work for case $k = 2$, since paths are not absolutely heavy in $\mathscr{P}_1$ (see Theorem 2.3).

(d) For each n and m, we construct a graph $G \in \mathscr{P}_3$ in which every n-cycle consists only of big vertices. Firstly we form the basic slice S. For n odd, let S be Q^-. For n even, $n \geqslant 8$, let S be K_4^-. For $n = 4$ or 6, let S be D^-.

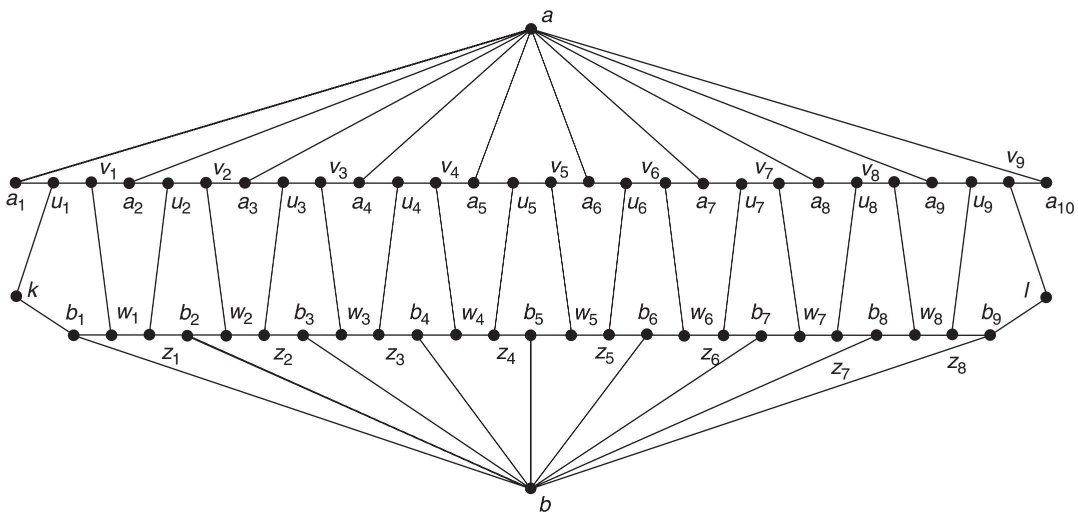

Fig. 2. Graph S for $n = 4, 6$ and $m = 10$.

Let M be the $(S, u, v; m)$-melon. Then G is obtained from the n-cycle C_n by identifying the endvertices of each its edge with the poles of a copy of M. It is easy to see that, in the resulting graph, the only n-cycle is the original one and all its vertices are big.

(e) Recall that the dual of the m-antiprism graph can be constructed from a $2m$-cycle $a_1b_1a_2b_2 \dots a_mb_m$ and two vertices a and b where a is adjacent to $a_1, \dots, a_m$ and b is adjacent to $b_1, \dots, b_m$. Let S be the graph obtained from the dual of the m-antiprism graph by deleting the 3-vertex a_1. Notice that b_1 and b_m are the only 2-vertices in S.

Take an n-cycle $C_n = x_1x_2 \dots x_n$ with $n \geqslant 3$ odd, and for each x_i take a new copy S^i of S. Next, identify x_i with a vertex b^i (the counterpart vertex of b in S^i). In addition, connect the vertex b^i_m with b^{i+1}_1, where these two vertices are counterparts of b_m and b_1 in S^i and S^{i+1}, respectively. The resulting graph $\widetilde{G}$ is a 3-connected planar graph with only two faces of odd size, one of them is a $5n$-cycle and the other one is the original n-cycle C_n. Moreover, all other faces of $\widetilde{G}$ are of size 4. Using the basis of the cycle space of $\widetilde{G}$ consisting of all its facial cycles, one can show that each odd cycle of $\widetilde{G}$ contains x_ix_{i+1} or $b^i_mb^{i+1}_1$ for each $i = 1, \dots, n$. This easily implies that the only n-cycle in $\widetilde{G}$ is $x_1x_2 \dots x_n$. Since each x_i is a big vertex in $\widetilde{G}$, we conclude that C_n with n odd is absolutely heavy in $\wp$.

For cycles of length 4 or 6, we will use the following construction: take two paths $P_a = a_1a_2 \dots a_m$, $P_b = b_1b_2 \dots b_{m-1}$ and two additional vertices a and b. Add edges aa_i for $i = 1, \dots, m$ and bb_j for $j = 1, \dots, m-1$. Next, subdivide each edge a_ia_{i+1}, $i = 1, \dots, m-1$ by two new vertices u_i and v_i such that a_iu_i, u_iv_i and v_ia_{i+1} are edges. Similarly, subdivide each edge b_jb_{j+1}, $j = 1, \dots, m-2$ by two new vertices w_j, z_j such that b_jw_j, w_jz_j, z_jb_{j+1} are edges. Add new edges v_jw_j and z_ju_{j+1} for $j = 1, \dots, m-2$. Then, add two new vertices k, l and edges kb_1, ku_1, lb_{m-1} and lv_{m-1}. In this way, we obtain a graph S with exactly one 10-face $aa_mv_{m-1}lb_{m-1}bb_1ku_1a_1$; all other faces of S are 5-faces (see Fig. 2 for $m = 10$).

Now, take an n-cycle $C_n = x_1x_2 \dots x_n$ with $n = 4$ or 6, and for each x_i take a new copy S^i of S. Next, identify x_i with the vertex b^i (the counterpart vertex in S^i of b). In addition, connect the vertex a^i_m with a^{i+1}_1 and identify the vertex l^i with the vertex k^{i+1}, where these four vertices are counterparts of a_m, a_1, k and l in S^i and S^{i+1}, respectively (see Fig. 3 for the case $n = 4$).

The resulting graph $\widetilde{G}$ is a 3-connected planar graph with only one face of size $3n$, only one face of size n (the original n-cycle C_n) and all remaining faces being pentagonal. By a routine check, one can easily check that no graph S^i, $i = 1, \dots, n$ contains an n-cycle; subsequently, in $\widetilde{G}$, there is no n-cycle C^* such that $E(C^*) \cap E(C_n) = \emptyset$.

Suppose first that $n = 4$ and there exists a 4-cycle $C^* \neq C_4$ such that $E(C^*) \cap E(C_4) \neq \emptyset$. Since $\widetilde{G}$ is simple, without loss of generality we can assume that $E(C^*) \cap E(C_4) = \{x_1x_2\}$ or $E(C^*) \cap E(C_4) = \{x_1x_2, x_2x_3\}$. In the former case $C^* = x_1x_2yz$, where y, z are distinct from x_3, x_4; now, observe that, in the graph $\widetilde{G} - E(C_4)$, the distance of x_1 and x_2 is 4, a contradiction. In the latter case $C^* = x_1x_2x_3y$, $y \neq x_4$; but, in the graph $\widetilde{G} - E(C_4)$, the shortest x_1–x_3-path that avoids x_2 and x_4 has length 10, a contradiction. Thus, the original C_4 is the only 4-cycle of $\widetilde{G}$ and it consists only of big vertices.

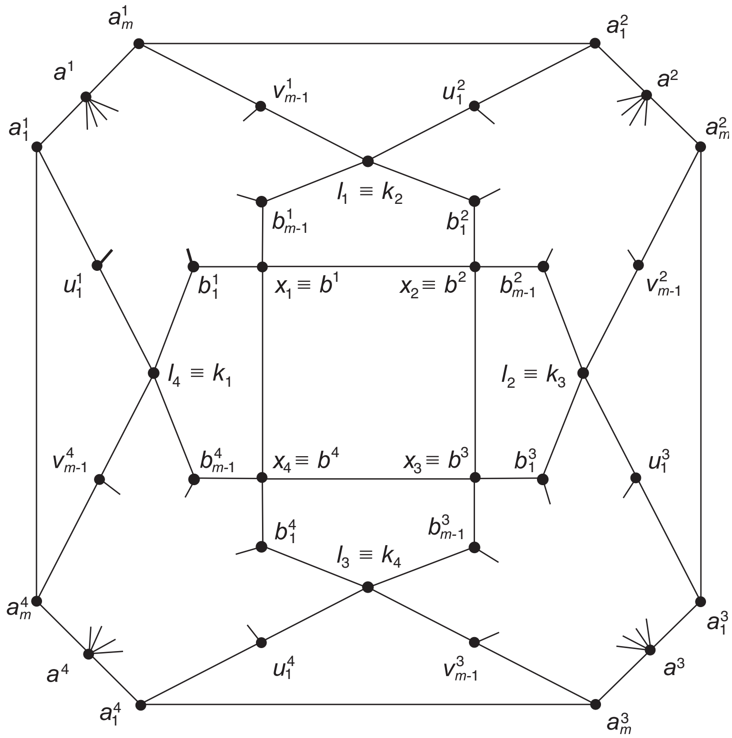

Fig. 3. Graph $\widetilde{G}$ for $n = 4$.

Next suppose that $n = 6$ and there exists a 6-cycle $C^* \neq C_6$ such that $E(C^*) \cap E(C_6) \neq \emptyset$. Observe that, in the graph $\widetilde{G} - E(C_6)$, the shortest x_1–x_2-path (x_1–x_3-path and x_1–x_4-path, respectively) avoiding any x_i has length 4 (length 10 and 13, respectively); this ensures that any cycle containing at most five edges of C_6 has length 5 or > 6, a contradiction. Hence, the original C_6 is again the only 6-cycle of $\widetilde{G}$ and it consists only of big vertices.

For the 8-cycle, the construction proceeds in the following way: for each $j = 1, \ldots, m$, take a new copy C^j of the configuration C of Fig. 4. Next, for $k = 1, \ldots, 5$ and $j = 1, \ldots, m-1$, connect the half-edges f_k^j with e_k^{j+1} (the counterpart half-edges in C^j and C^{j+1} of f_k and e_k) and identify all vertices x^l, $l = 1, \ldots, m$ (the counterparts in C^l of x); let x be the vertex resulted from this identification. The configuration S obtained in this way consists of 5- and 6-gons and it contains half-edges $\widehat{e_k} = e_k^1$, $\widehat{f_k} = f_k^m$, $k = 1, \ldots, 5$ lying in the "outerface".

Now, take an 8-cycle $C_8 = x_1 x_2 \ldots x_8$ and for each x_i take a new copy S^i of S. Next, identify x_i with the vertex x^i (the counterpart vertex in S^i of x). In addition, connect the half-edges $\widehat{f_k^i}$ with the half-edges $\widehat{e}_k^{i+1}$ for each $k = 1, \ldots, 5$ (index i is taken modulo 8), as depicted in Fig. 5. The resulting plane 3-connected graph $\widetilde{G}$ consists of 5-, 6- and 7-faces, exactly one "big" face and exactly one 8-face (which is the only 8-cycle in this graph), all of which vertices are big. To show this, we use the arguments similar to the ones for the 4-cycle:

- The configuration C of Fig. 4 contains no 8-cycle: observe first that, in the plane graph C' obtained from C by removing its half-edges, there is no 8-cycle meeting an edge which has at least one endvertex incident with the outerface of C' (this can be verified by hand). Remove all these edges from C' (and, also, the isolated vertices resulted from this removal) and repeat the argument for the remaining graph C''; again, there is no 8-cycle meeting an edge having an endvertex incident with the outerface of C''. Repeating this process once more results in a path; thus, no 8-cycle is present in C.
- The configuration S composed of the copies of C contains no 8-cycle: if there is an 8-cycle in S, then, by the previous argument, it meets at least two edges which interconnect two consecutive copies of C in S (i.e. the dashed edges in

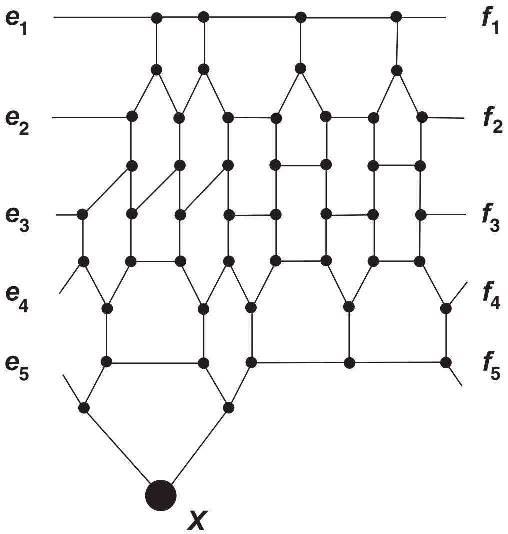

Fig. 4. Configuration C.

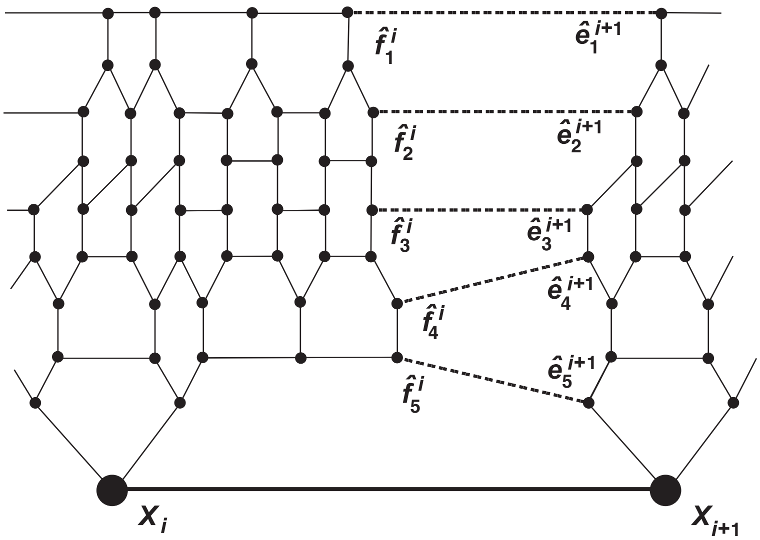

Fig. 5. Interconnection of two configurations S^i, S^{i+1} for $n = 8$.

Fig. 5). By examining all $\binom{5}{2} = 10$ pairs of those edges between any two consecutive copies of C in S (note that taking two edges from different interconnections always results in a longer cycle), we obtain that each cycle meeting the selected pair has length 5,6 or at least 9, but never 8.

- Therefore, if there is an 8-cycle in $\widetilde{G}$, then it contains some edges of the original C_8. Now, observe that, in $\widetilde{G} - E(C_8)$, each x_i–x_{i+1}-path has length at least 6 and each x_i–x_j-path, $j \neq i + 1$ (indices taken modulo 8), has length at least 12; this again gives that each cycle meeting some (but not all) of the edges of C_8 consists of more than 8 edges.

For the 10-cycle, the construction of the configuration S is similar as for the 8-cycle (with the configuration $\widehat{C}$ of Fig. 6 instead of C). Also, the construction of the graph $\widetilde{G}$ is similar; just, when connecting the half-edges of S^i and S^{i+1}, additional vertices have to be introduced, see Fig. 7. The graph $\widetilde{G}$ consists of 4-, 7- and 9-faces, exactly one "big" face and exactly one 10-face (which is the only 10-cycle in this graph), all of which vertices are big. To see it, we give

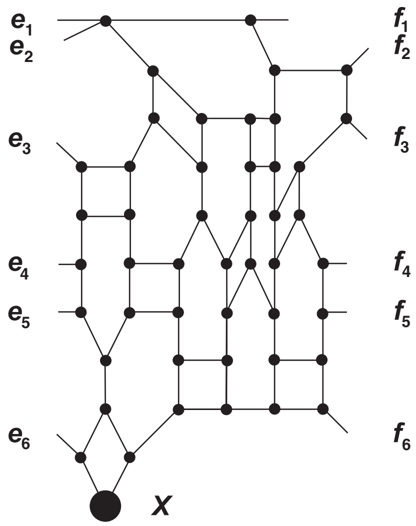

Fig. 6. Configuration $\widehat{C}$.

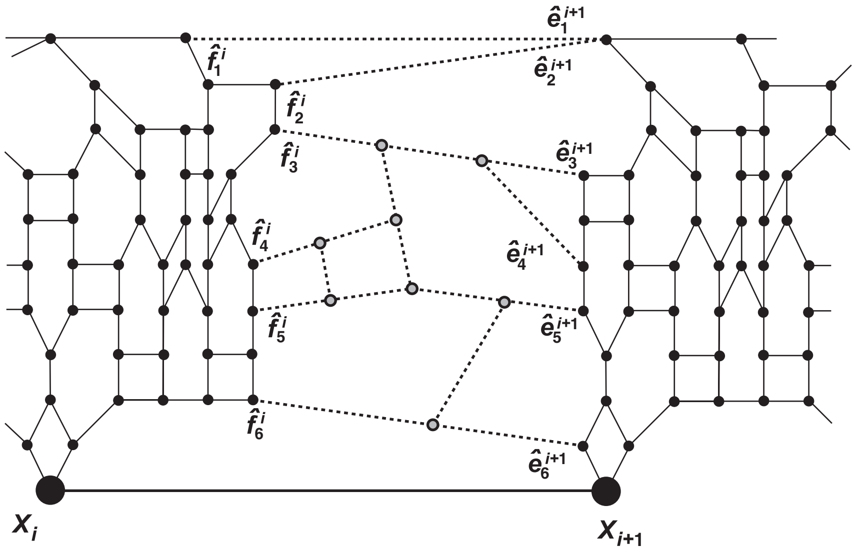

Fig. 7. Interconnection of two configurations S^i, S^{i+1} for $n=10$.

the following three arguments:

- The configuration $\widehat{C}$ contains no 10-cycle: remove first the half-edges of $\widehat{C}$ obtaining thereby the plane graph $\widehat{C}'$. Each its cycle which has at least one vertex common with the outerface of $\widehat{C}'$ is of length 4, 7, 9 or at least 11. After removing the outerface edges and all edges incident with them (together with isolated vertices) we obtain a plane graph $\widehat{C}''$ on 22 vertices whose interior (i.e. complement of outerface) consists of three 4-faces and three 7-faces. Now it is easy to check that, in $\widehat{C}''$, the 10-cycle is missing.
- The configuration S^i and the configuration S formed by interconnections of S^i, $i=1,\ldots,10$ contains no 10-cycle: it is easy to check that, when connecting two copies of $\widehat{C}$ consecutively, no 10-cycle can appear (it is enough to verify $\binom{6}{2}$ pairs of edges formed by joining the half-edges e_i with f_i, $i=1,\ldots,6$). Considering the dotted edges

interconnecting two copies S^i, S^{i+1}, observe first that no 10-cycle can be formed taking only the faces of the set $\bar{F}_i$ (which is the set of 10 faces each of them being incident with a dotted edge). Thus, if there is a 10-cycle, it necessarily has to meet some of these 10 faces and some other faces which are part of S^i or S^{i+1}. Since, in the interior of $\widehat{C}$, there are only 4- and 7-faces and each its 4-face is incident only with 7-faces, it is enough to check the faces incident with the faces of the set $\bar{F}_i$. Again, a routine check gives that no 10-cycle can be formed.
- By two arguments above, any 10-cycle of $\widetilde{G}$ has to meet some edges of the original C_{10}. But, in $\widetilde{G} - E(C_{10})$, each x_i–x_{i+1}-path has length either 8 or at least 11, and each x_i–x_j-path, $j \neq i + 1$, has length at least 16; thus, each cycle meeting some (but not all) of the edges of the original C_{10} consists either of 9 or at least 12 edges. □

5. Some problems

One possible further work is to study the gravity of the paths in $\mathscr{P}_3$, $\mathscr{P}_4$ or $\mathscr{P}_5$ as well as resolving the few left cases from Section 2.3. However, we conclude the paper by posing several problems about the gravity of the graphs in $\wp$.

Problem 5.1. *For a given integer n, are there infinitely many n-heavy graphs in the family* $\wp$?

Regarding the above problem, consider all k-stars with a quadrangle attached to one leaf. In $\wp$, each such a graph has gravity $\geqslant 2$ (see the dual of m-antiprism). Regarding the next problem, notice that by Theorem 4.1(e), the open cases are the even cycles of length $n \geqslant 12$.

Problem 5.2. *Is every cycle absolutely heavy in the family* $\wp$?

An important example of a family of plane graphs with infinite set of light graphs is $\wp$ (see [3]), where $\mathscr{L}(\wp) = \{P_k, k \geqslant 1\}$. Hence, $\overline{\mathscr{L}(\wp)} = \{P_k, k \geqslant 1\} \cup \{C_k, k \geqslant 1\} \cup \{S_3\}$ and by Theorem 4.1(e), $\mathscr{A}\mathscr{L}(\wp) \subseteq \{S_3\} \cup \{C_k, k \geqslant 12 \text{ even}\}$.

Problem 5.3. *Find all almost-light graphs and find all* 1*-heavy graphs in the family* $\wp$.

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